RESEARCHERID
THOMSON REUTERS

[Gulzar * *et al.*, 6(9): September, 2017]
IC™ Value: 3.00

ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

NUMBER OF ZEROS OF A POLYNOMIAL IN A REGION

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DOI: 10.5281/zenodo.892064

ABSTRACT

The aim of this paper is to correct the bounds for the zeros of certain polynomials recently proved by K.A.Kareem and A.A.Mogbademu [3]

Mathematics Subject Classification (2010): 30C10, 30C15.

KEYWORDS: Coefficient, Open Disc, Polynomial, Zeros

I. INTRODUCTION

In the framework of Enestrom-Kakeya theorem([4],[5]) which states that "all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with $a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0$ lie in $|z| \leq 1$", K.A.Kareem and A.A.Mogbademu [3] recently, while generalizing some results of Gardner and Shields [1] on polynomials with certain monotonicity conditions on their coefficients, claim to have proved the following results:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some R>0,

$0 \leq \mu < 1, 0 < \rho \leq 1, 0 \leq k \leq n,$

$$0 < \rho|a_0| \leq R|a_1| \leq R^2|a_2| \leq .... \leq R^{k-1}|a_{k-1}| \leq R^k|a_k|$$
$$\geq R^{k+1}|a_{k+1}| \geq ...... \geq R^{n-1}|a_{n+1}| \geq (R-\mu)R^{n-1}|a_n|,$$

and $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ for $1 \leq j \leq n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \leq \delta R$ is less than

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{M}{|a_0|}$$

where

$$M = |a_0|R[\frac{1}{\rho} + \frac{\mu}{\rho|a_0|} - \cos\alpha - \sin\alpha] + 2|a_k|R^{k+1}\cos\alpha +$$
$$+|a_n|R^{n+1}[1+\frac{\mu}{|a_n|} - \cos\alpha + \sin\alpha] + 2\sin\alpha \sum_{j=0}^{n-1}|a_j|R^{j+1}.$$

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$ for $0 \leq j \leq n$. Suppose that for some R>0, $0 \leq \mu < 1, 0 < \rho \leq 1, 0 \leq k \leq n,$

$$0 \neq \rho\alpha_0 \leq R\alpha_1 \leq R^2\alpha_2 \leq \dots\dots\dots \leq R^{k-1}\alpha_{k-1} \leq R^k\alpha_k$$
$$\geq R^{k+1}\alpha_{k+1} \geq \dots\dots \geq R^{n-1}\alpha_{n+1} \geq (R-\mu)R^{n-1}\alpha_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z| \leq \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = (\frac{1}{\rho}|\alpha_0-\mu|-\alpha_0)R + (|\alpha_n-\mu|-\alpha_n)R^{n+1} + \mu R(1+R^n)$$
$$+2\alpha_k R^{k+1} + 2\sum_{j=0}^{n}|\beta_j|R^{j+1}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$ for $0\leq j\leq n$. Suppose that for some R>0, $0\leq\mu<1, 0\leq\lambda<1, 0<\rho_1\leq 1, 0<\rho_2\leq 1$ and for some $0\leq k\leq n, 0\leq l\leq n$,

$$0 \neq \rho_1\alpha_0 \leq R\alpha_1 \leq R^2\alpha_2 \leq \dots\dots\dots \leq R^{k-1}\alpha_{k-1} \leq R^k\alpha_k$$
$$\geq R^{k+1}\alpha_{k+1} \geq \dots\dots \geq R^{n-1}\alpha_{n+1} \geq (R-\mu)R^{n-1}\alpha_n$$

and

$$\rho_2\beta_0 \leq R\beta_1 \leq R^2\beta_2 \leq \dots\dots\dots \leq R^{l-1}\beta_{l-1} \leq R^l\beta_l$$
$$\geq R^{l+1}\beta_{l+1} \geq \dots\dots \geq R^{n-1}\beta_{n+1} \geq (R-\mu)R^{n-1}\beta_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in the disc $|z| \leq \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = (\frac{1}{\rho_1}|\alpha_0-\mu|-\alpha_0)R + (\frac{1}{\rho_2}|\beta_0-\lambda|-\beta_0)R + (\mu+\lambda)R(1+R^n)$$
$$+2\alpha_k R^{k+1} + (|\alpha_n-\mu|-\alpha_n)R^{n+1} + (|\beta_n-\lambda|-\beta_n)R^{n+1} + 2\beta_l R^{l+1}.$$

II. MAIN RESULTS

Unfortunatelely, there are various mistakes in the proofs of these theorems and the bounds obtained for the moduli of the zeros are not correct. In this paper we give the correct bounds for the zeros of the polynomials in the above mentioned theorems and prove

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some R>0,

$0\leq\mu<1, 0<\rho\leq 1, 0\leq k\leq n,$

$$0 < \rho|a_0| \leq R|a_1| \leq R^2|a_2| \leq \dots\dots\dots \leq R^{k-1}|a_{k-1}| \leq R^k|a_k|$$
$$\geq R^{k+1}|a_{k+1}| \geq \dots\dots \geq R^{n-1}|a_{n+1}| \geq (R-\mu)R^{n-1}|a_n|,$$

and $\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}$ for $1 \le j \le n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$\begin{aligned} M &= 2|a_0|R - \rho|a_0|R(1-\cos\alpha-\sin\alpha) + 2|a_k|R^{k+1}\cos\alpha \\ &\quad + |a_n|R^{n+1}(1-\cos\alpha+\sin\alpha) + \mu|a_n|R^n(1+\cos\alpha-\sin\alpha) \\ &\quad + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}. \end{aligned}$$

Taking R=1 in Theorem 1, we get the following

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some $0 \le \mu < 1, 0 < \rho \le 1, 0 \le k \le n,$

$$\begin{aligned} 0 < \rho|a_0| \le |a_1| \le |a_2| \le \dots\dots\dots \le |a_{k-1}| \le |a_k| \\ \ge |a_{k+1}| \ge \dots\dots \ge |a_{n+1}| \ge (1-\mu)|a_n|, \end{aligned}$$

and $\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}$ for $1 \le j \le n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$\begin{aligned} M &= 2|a_0| - \rho|a_0|(1-\cos\alpha-\sin\alpha) + 2|a_k|\cos\alpha \\ &\quad + |a_n|(1-\cos\alpha+\sin\alpha) + \mu|a_n|(1+\cos\alpha-\sin\alpha) \\ &\quad + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|. \end{aligned}$$

For $\mu = 0, \rho = 1$, we get the following result of Gardner and Sheilds [1] from Theorem 1.

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n where for some R>0, $0 \le k \le n,$

$$\begin{aligned} 0 < |a_0| \le R|a_1| \le R^2|a_2| \le \dots\dots\dots \le R^{k-1}|a_{k-1}| \le R^k|a_k| \\ \ge R^{k+1}|a_{k+1}| \ge \dots\dots \ge R^{n-1}|a_{n+1}| \ge R^n|a_n|, \end{aligned}$$

and $\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2}$ for $1 \le j \le n$ and for some real α and β. Then for $0 < \delta < 1$ the number of zeros of P(z) in $|z| \le \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = |a_0|R(1-\cos\alpha-\sin\alpha)+2|a_k|R^{k+1}\cos\alpha$$
$$+|a_n|R^{n+1}(1-\cos\alpha+\sin\alpha)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}.$$

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n where $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $0\le j\le n$. Suppose that for some R>0, $0\le\mu<1, 0<\rho\le 1, 0\le k\le n,$

$$0\ne\rho\alpha_0\le R\alpha_1\le R^2\alpha_2\le ....\le R^{k-1}\alpha_{k-1}\le R^k\alpha_k$$
$$\ge R^{k+1}\alpha_{k+1}\ge ......\ge R^{n-1}\alpha_{n+1}\ge(R-\mu)R^{n-1}\alpha_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = 2|\alpha_0|R-\rho(\alpha_0+|\alpha_0|)R+2\alpha_kR^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu(\alpha_n+|\alpha_n|)R^n$$
$$+2\sum_{j=0}^{n}|\beta_j|R^{j+1}\quad.$$

Taking R=1 in Theorem 2, we get the following

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n where $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $0\le j\le n$. Suppose that for some $0\le\mu<1, 0<\rho\le 1, 0\le k\le n,$

$$0\ne\rho\alpha_0\le\alpha_1\le\alpha_2\le ....\le\alpha_{k-1}\le\alpha_k$$
$$\ge\alpha_{k+1}\ge ......\ge\alpha_{n+1}\ge(1-\mu)\alpha_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z|\le\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = 2|\alpha_0|-\rho(\alpha_0+|\alpha_0|)+2\alpha_k+(|\alpha_n|-\alpha_n)+\mu(\alpha_n+|\alpha_n|)$$
$$+2\sum_{j=0}^{n}|\beta_j|\quad.$$

Taking $\mu=0,\rho=1$, we get the following result of Gardner and Sheilds [1] from Theorem 2.

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n where $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $0\le j\le n$. Suppose that for some R>0, $0\le k\le n,$

$$0\ne\alpha_0\le R\alpha_1\le R^2\alpha_2\le ....\le R^{k-1}\alpha_{k-1}\le R^k\alpha_k$$

$$\geq R^{k+1}\alpha_{k+1} \geq ...... \geq R^{n-1}\alpha_{n+1} \geq R^{n}\alpha_{n}.$$

Then for $0<\delta<1$ the number of zeros of P(z) in $|z| \leq \delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M=|\alpha_0|-\alpha_0)R+2\alpha_k R^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1} \quad +2\sum_{j=0}^{n}|\beta_j|R^{j+1} \quad .$$

Theorem 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n where $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $0\leq j\leq n$. Suppose that for some R>0, $0\leq\mu<1, 0\leq\lambda<1, 0<\rho_1\leq 1, 0<\rho_2\leq 1$ and for some $0\leq k\leq n, 0\leq l\leq n,$

$$0\neq\rho_1\alpha_0\leq R\alpha_1\leq R^2\alpha_2\leq .... \leq R^{k-1}\alpha_{k-1}\leq R^k\alpha_k$$
$$\geq R^{k+1}\alpha_{k+1}\geq ......\geq R^{n-1}\alpha_{n+1}\geq (R-\mu)R^{n-1}\alpha_n$$

and

$$\rho_2\beta_0\leq R\beta_1\leq R^2\beta_2\leq ....\leq R^{l-1}\beta_{l-1}\leq R^l\beta_l$$
$$\geq R^{l+1}\beta_{l+1}\geq ......\geq R^{n-1}\beta_{n+1}\geq (R-\lambda)R^{n-1}\beta_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in the disc $|z|\leq\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M=2|\alpha_0|R-\rho_1 R(|\alpha_0|+\alpha_0)+2\alpha_k R^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu(\alpha_n+|\alpha_n|)R^n$$
$$+2|\beta_0|R-\rho_2 R(|\beta_0|+\beta_0)+2\beta_l R^{l+1}+(|\beta_n|-\beta_n)R^{n+1}+\lambda(\beta_n+|\beta_n|)R^n.$$

Taking $\rho_1=\rho_2=1, \lambda=\mu=0$ in Theorem 3, we get the following result due to Gardner and Sheilds [1]:

Corollary 5: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n where $\text{Re}(a_j)=\alpha_j$ and $\text{Im}(a_j)=\beta_j$ for $0\leq j\leq n$. Suppose that for some R>0, $0<\rho\leq 1$, and for some $0\leq k\leq n,$

$$0\neq\alpha_0\leq R\alpha_1\leq R^2\alpha_2\leq ....\leq R^{k-1}\alpha_{k-1}\leq R^k\alpha_k$$
$$\geq R^{k+1}\alpha_{k+1}\geq ......\geq R^{n-1}\alpha_{n+1}\geq R^n\alpha_n$$

and

$$\beta_0\leq R\beta_1\leq R^2\beta_2\leq ....\leq R^{l-1}\beta_{l-1}\leq R^l\beta_l$$
$$\geq R^{l+1}\beta_{l+1}\geq ......\geq R^{n-1}\beta_{n+1}\geq R^n\beta_n.$$

Then for $0<\delta<1$ the number of zeros of P(z) in the disc $|z|\leq\delta R$ is less than

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$$

where

$$M = |\alpha_0| - \alpha_0)R + 2\alpha_k R^{k+1} + (|\alpha_n| - \alpha_n)R^{n+1}$$
$$+ (|\beta_0| - \beta_0)R + 2\beta_l R^{l+1} + (|\beta_n| - \beta_n)R^{n+1}.$$

For other different values of the parameters , we get many other interesting results from the above results.

III. LEMMAS

For the proof s of the above results, we make use of the following lemma which is due to Govil and Rahman [2]:

Lemma1: For any two complex numbers z_1, z_2 such that $|z_1| \geq |z_2|$ and for some real α, β , $|\arg z_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$, we have

$$|z_1 - z_2| \leq (|z_1| - |z_2|)\cos\alpha + (|z_1| + |z_2|)\sin\alpha .$$

Lemma 1 is due to Govil and Rahman [2].

Lemma2: Let f(z) be analytic , $|f(z)| \leq M$ for $|z| \leq R$ and $f(0) \neq 0$. Then the number of zeros of f(z) in $|z| \leq \delta R$ where $0 < \delta < 1$ is less than or equal to $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$.

For Lemma 2 see [6]).

IV. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

F(z)=(R-z)P(z)

$$= (R - z)(a_0 + a_1 z + a_2 z^2 + ... + a_{k-1}z^{k-1} + a_k z^k + a_{k+1}z^{k+1} + ... + a_{n-1}z^{n-1} + a_n z^n)$$
$$= a_0 R + (a_1 R - a_0)z + (a_2 R - a_1)z^2 + ... + (a_{k-1}R - a_{k-2})z^{k-1} + (a_k R - a_{k-1})z^k$$
$$+ (a_{k+1}R - a_k)z^{k+1} + ... + (a_{n-1}R - a_{n-2})z^{n-1} + (a_n R - a_{n-1})z^n - a_n z^{n+1}.$$

For $|z| \leq R$, we have, by using the hypothesis and the Lemma,

$$|F(z)| \leq |a_0|R + |a_1 R - \rho a_0 + \rho a_0 - a_0|R + |a_2 R - a_1|R^2 + ... + |a_{k-1}R - a_{k-2}|R^{k-1} + |a_k R - a_{k-1}|R^k$$
$$+ |a_{k+1}R - a_k|R^{k+1} + ... + |a_{n-1}R - a_{n-2}|R^{n-1} + |a_n R - \mu a_n + \mu a_n - a_{n-1}|R^n + |a_n|R^{n+1}$$
$$\leq |a_0|R + [(|a_1|R - \rho|a_0|)\cos\alpha + (|a_1|R + \rho|a_0|)\sin\alpha]R + (1 - \rho)|a_0|R$$
$$+ [(|a_2|R - |a_1|)\cos\alpha + (|a_2|R + |a_1|)\sin\alpha]R^2 + ...$$

$$+[(|a_{k-1}|R-|a_{k-2}|)\cos\alpha+(|a_{k-1}|R+|a_{k-2}|)\sin\alpha]R^{k-1}$$
$$+[(|a_k|R-|a_{k-1}|)\cos\alpha+(|a_k|R+|a_{k-1}|)\sin\alpha]R^{k}$$
$$+[(|a_k|-|a_{k+1}|R)\cos\alpha+(|a_k|+|a_{k+1}|R)\sin\alpha]R^{k+1}+......$$
$$+[(|a_{n-2}|-|a_{n-1}|R)\cos\alpha+(|a_{n-2}|+|a_{n-1}|R)\sin\alpha]R^{n-1}$$
$$+[\{|a_{n-1}|-(R-\mu)|a_n|\}\cos\alpha+\{|a_{n-1}|+(R-\mu)|a_n|\}\sin\alpha]R^{n}$$
$$+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$=|a_0|R+(1-\rho)|a_0|R-\rho|a_0|R\cos\alpha+\rho|a_0|R\sin\alpha+2|a_k|R^{k+1}\cos\alpha-|a_n|R^{n+1}\cos\alpha$$
$$+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$=2|a_0|R-\rho|a_0|-\rho|a_0|R\cos\alpha-\rho|a_0|R\sin\alpha+2\rho|a_0|R\sin\alpha+2|a_k|R^{k+1}\cos\alpha$$
$$-|a_n|R^{n+1}\cos\alpha+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$\le 2|a_0|R-\rho|a_0|R(1-\cos\alpha-\sin\alpha)+2|a_0|R\sin\alpha+2|a_k|R^{k+1}\cos\alpha$$
$$-|a_n|R^{n+1}\cos\alpha+|a_n|R^{n+1}\sin\alpha+\mu|a_n|R^n\cos\alpha-\mu|a_n|R^n\sin\alpha+\mu|a_n|R^n+|a_n|R^{n+1}$$
$$+2\sin\alpha\sum_{j=1}^{n-1}|a_j|R^{j+1}$$
$$=2|a_0|R-\rho|a_0|R(1-\cos\alpha-\sin\alpha)+2|a_k|R^{k+1}\cos\alpha$$
$$+|a_n|R^{n+1}(1-\cos\alpha+\sin\alpha)+\mu|a_n|R^n(1+\cos\alpha-\sin\alpha)$$
$$+2\sin\alpha\sum_{j=0}^{n-1}|a_j|R^{j+1}$$
$$=M\ .$$

Since F(z) is also analytic for $|z|\le R$, F(0) $\neq 0$, it follows by the lemma that the number of zeros of F(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}\,.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that the number of zeros of P(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}\,.$$

That proves theorem 1.

Proof of Theorem 2: Consider the polynomial

F(z)=(R-z)P(z)

$$=(R-z)(a_0+a_1z+a_2z^2+.....+a_{k-1}z^{k-1}+a_kz^k+a_{k+1}z^{k+1}+......+a_{n-1}z^{n-1}+a_nz^n)$$

$$=a_0R+(a_1R-a_0)z+(a_2R-a_1)z^2+......+(a_{k-1}R-a_{k-2})z^{k-1}+(a_kR-a_{k-1})z^k$$
$$+(a_{k+1}R-a_k)z^{k+1}+......+(a_{n-1}R-a_{n-2})z^{n-1}+(a_nR-a_{n-1})z^n-a_nz^{n+1}.$$

$$=\alpha_0R+(\alpha_1R-\alpha_0)z+(\alpha_2R-\alpha_1)z^2+......+(\alpha_{k-1}R-\alpha_{k-2})z^{k-1}+(\alpha_kR-\alpha_{k-1})z^k$$
$$+(\alpha_{k+1}R-\alpha_k)z^{k+1}+......+(\alpha_{n-1}R-\alpha_{n-2})z^{n-1}+(\alpha_nR-\alpha_{n-1})z^n-\alpha_nz^{n+1}$$
$$+i[\beta_0R+(\beta_1R-\beta_0)z+(\beta_2R-\beta_1)z^2+......+(\beta_{k-1}R-\beta_{k-2})z^{k-1}$$
$$+(\beta_kR-\beta_{k-1})z^k+(\beta_{k+1}R-\beta_k)z^{k+1}+......+(\beta_{n-1}R-\beta_{n-2})z^{n-1}$$
$$+(\beta_nR-\beta_{n-1})z^n-\beta_nz^{n+1}]$$

For $|z|\le R$, we have, by using the hypothesis ,

$$|F(z)|\le|\alpha_0|R+|\alpha_1R-\rho\alpha_0+\rho\alpha_0-\alpha_0|R+|\alpha_2R-\alpha_1|R^2+......+|\alpha_{k-1}R-\alpha_{k-2}|R^{k-1}+|\alpha_kR-\alpha_{k-1}|R^k$$

$$+|\alpha_{k+1}R-\alpha_k|R^{k+1}+......+|\alpha_{n-1}R-\alpha_{n-2}|R^{n-1}+|\alpha_nR-\mu\alpha_n+\mu\alpha_n-\alpha_{n-1}|R^n+|\alpha_n|R^{n+1}$$
$$+|\beta_0|R+|\beta_1R-\beta_0|R+|\beta_2R-\beta_1|R^2+......+|\beta_{n-1}R-\beta_{n-2}|R^{n-1}+|\beta_nR-\beta_{n-1}|R^n+|\beta_n|R^{n+1}$$
$$\le|\alpha_0|R+\alpha_1R^2-\rho\alpha_0R+(1-\rho)|\alpha_0|R+\alpha_2R^3-\alpha_1R^2+\alpha_3R^4-\alpha_2R^3+......$$
$$+\alpha_{k-1}R^k-\alpha_{k-2}R^{k-1}+\alpha_kR^{k+1}-\alpha_{k-1}R^k+\alpha_kR^{k+1}-\alpha_{k+1}R^{k+2}+......$$
$$+\alpha_{n-2}R^{n-1}-\alpha_{n-1}R^n+\alpha_{n-1}R^n-(R-\mu)\alpha_nR^n+\mu|\alpha_n|R^n+|\alpha_n|R^{n+1}$$
$$+|\beta_0|R+|\beta_1|R^2+|\beta_0|R+|\beta_2|R^3+|\beta_1|R^2+......+|\beta_{n-1}|R^n+|\beta_{n-2}|R^{n-1}$$
$$+|\beta_n|R^{n+1}+|\beta_{n-1}|R^n+|\beta_n|R^{n+1}$$
$$\le2|\alpha_0|R-\rho(\alpha_0+|\alpha_0|)R+2\alpha_kR^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu(\alpha_n+|\alpha_n|)R^n$$
$$+2\sum_{j=0}^{n}|\beta_j|R^{j+1}$$
$$=M\ .$$

Since F(z) is also analytic for $|z|\le R$, F(0) $\neq 0$, it follows by the lemma that the number of zeros of F(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that the number of zeros of P(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}.$$

That proves theorem 2.

ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

Proof of Theorem 3: Consider the polynomial

F(z)=(R-z)P(z)

$$
\begin{aligned}
&=(R-z)(a_0+a_1z+a_2z^2+.....+a_{k-1}z^{k-1}+a_kz^k+a_{k+1}z^{k+1}+......+a_{n-1}z^{n-1}+a_nz^n)\\
&=a_0R+(a_1R-a_0)z+(a_2R-a_1)z^2+......+(a_{k-1}R-a_{k-2})z^{k-1}+(a_kR-a_{k-1})z^k\\
&\quad+(a_{k+1}R-a_k)z^{k+1}+......+(a_{n-1}R-a_{n-2})z^{n-1}+(a_nR-a_{n-1})z^n-a_nz^{n+1}\\
&=\alpha_0R+(\alpha_1R-\alpha_0)z+(\alpha_2R-\alpha_1)z^2+......+(\alpha_{k-1}R-\alpha_{k-2})z^{k-1}+(\alpha_kR-\alpha_{k-1})z^k\\
&\quad+(\alpha_{k+1}R-\alpha_k)z^{k+1}+......+(\alpha_{n-1}R-\alpha_{n-2})z^{n-1}+(\alpha_nR-\alpha_{n-1})z^n\\
&\quad-\alpha_nz^{n+1}+i[\beta_0R+(\beta_1R-\beta_0)z+(\beta_2R-\beta_1)z^2+......+(\beta_{k-1}R-\beta_{k-2})z^{k-1}\\
&\quad+(\beta_kR-\beta_{k-1})z^k+(\beta_{k+1}R-\beta_k)z^{k+1}+......+(\beta_{n-1}R-\beta_{n-2})z^{n-1}\\
&\quad+(\beta_nR-\beta_{n-1})z^n-\beta_nz^{n+1}]
\end{aligned}
$$

For $|z|\le R$, we have, by using the hypothesis

$$
\begin{aligned}
|F(z)|&\le|\alpha_0|R+|\alpha_1R-\rho_1\alpha_0+\rho_1\alpha_0-\alpha_0|R+|\alpha_2R-\alpha_1|R^2+......+|\alpha_{k-1}R-\alpha_{k-2}|R^{k-1}+|\alpha_kR-\alpha_{k-1}|R^k\\
&\quad+|\alpha_{k+1}R-\alpha_k|R^{k+1}+......+|\alpha_{n-1}R-\alpha_{n-2}|)R^{n-1}+|\alpha_nR-\mu\alpha_n+\mu\alpha_n-\alpha_{n-1}|R^n\\
&\quad+|\alpha_n|R^{n+1}+|\beta_0|R+|\beta_1R-\rho_2\beta_0+\rho_2\beta_0-\beta_0|R+|\beta_2R-\beta_1|R^2+.......\\
&\quad+|\beta_{l-1}R-\beta_{l-2}|R^{l-1}+|\beta_lR-\beta_{l-1}|R^l+......+|\beta_{n-1}R-\beta_{n-2}|)R^{n-1}\\
&\quad+|\beta_nR-\lambda\beta_n+\lambda\beta_n-\beta_{n-1}|R^n+|\beta_n|R^{n+1}\\
&\le|\alpha_0|R+(\alpha_1R-\rho_1\alpha_0)R+(1-\rho_1)|\alpha_0|R+(\alpha_2R-\alpha_1)R^2+......+(\alpha_{k-1}R-\alpha_{k-2})R^{k-1}\\
&\quad+(\alpha_kR-\alpha_{k-1})R^k+(\alpha_k-\alpha_{k+1}R)R^{k+1}+......+(\alpha_{n-2}-\alpha_{n-1}R)R^{n-1}\\
&\quad+[\{\alpha_{n-1}-(R-\mu)\alpha_n\}]R^n+\mu|\alpha_n|R^n+|\alpha_n|R^{n+1}+|\beta_0|R+(\beta_1R-\rho_2\beta_0)R\\
&\quad+(1-\rho_2)|\beta_0|R+(\beta_2R-\beta_1)R^2+......+(\beta_{l-1}R-\beta_{l-2})R^{l-1}\\
&\quad+(\beta_lR-\beta_{l-1})R^l+(\beta_l-\beta_{l+1}R)R^{l+1}+......+(\beta_{n-2}-\beta_{n-1}R)R^{n-1}\\
&\quad+[\{\beta_{n-1}-(R-\lambda)\beta_n\}]R^n+\lambda|\alpha_n|R^n+|\beta_n|R^{n+1}\\
&\le 2|\alpha_0|R-\rho_1R(|\alpha_0|+\alpha_0)+2\alpha_kR^{k+1}+(|\alpha_n|-\alpha_n)R^{n+1}+\mu(\alpha_n+|\alpha_n|)R^n\\
&\quad+2|\beta_0|R-\rho_2R(|\beta_0|+\beta_0)+2\beta_lR^{l+1}+(|\beta_n|-\beta_n)R^{n+1}+\lambda(\beta_n+|\beta_n|)R^n\\
&=M
\end{aligned}
$$

Since F(z) is also analytic for $|z|\le R$, F(0) $\neq 0$, it follows by Lemma 2 that the number of zeros of F(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that the number of zeros of P(z) in $|z|\le\delta R, 0<\delta<1$ is less than or equal to

ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|F(0)|}=\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|R}.$$

That proves theorem 3.

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CITE AN ARTICLE

Gulzar, M. H., Wani, A., & Hussain, I. (2017). NUMBER OF ZEROS OF A POLYNOMIAL IN A REGION. *INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY*, *6*(9), 346-355. Retrieved September 15, 2017.